Mathematics: analysis and approaches
Higher level
Paper 2

Name

worked solutions_v2

Date: ____________________

2 hours

SOLUTION KEY

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written in the answer boxes provided.
- Section B: answer all questions on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

exam: 9 pages

Section A (55 marks)

1. [Maximum mark: 6]

A discrete random variable has a probability distribution given in the following table.

x	3	4	5	6	7	8
$\mathrm{P}(X=x)$	0.08	a	0.30	b	0.24	0.12

(a) Given that the expected value of X is 5.62, find the value of a and the value of b. [4]

(b) Calculate the variance of X. [2]

Solution:

(a) $\mathrm{E}(X)=\sum x\cdot\mathrm{P}(X=x)=3(0.08)+4a+5(0.3)+6b+7(0.24)+8(0.12)=5.62 \quad\Rightarrow\quad 4a+6b=1.24$

and $\mathrm{P}(X=x)=0.08+a+0.3+b+0.24+0.12=1 \quad\Rightarrow\quad a+b=0.26$

$$\begin{cases}4a+6b=1.24\\ a+b=0.26\end{cases} \quad\Rightarrow\quad a=0.16,\, b=0.1$$

(b) $\mathrm{Var}(X)=\sum x^2\cdot\mathrm{P}(X=x)-\left[\mathrm{E}(X)\right]^2$

$$=3^2(0.08)+4^2(0.16)+5^2(0.3)+6^2(0.1)+7^2(0.24)+8^2(0.12)-5.62^2$$

$\mathrm{Var}(X)=2.2356\approx 2.24$

2. [Maximum mark: 6]

Events A and B are such that $\mathrm{P}(A\cup B)=0.9$, $\mathrm{P}(A\cap B)=0.45$ and $\mathrm{P}(A|B)=0.75$.

(a) Find $\mathrm{P}(B)$. [2]

(b) Find $\mathrm{P}(A)$. [2]

(c) Hence, show that events A and B are independent. [2]

Solution:

(a) $\mathrm{P}(A|B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)} \quad\Rightarrow\quad 0.75=\dfrac{0.45}{\mathrm{P}(B)} \quad\Rightarrow\quad \mathrm{P}(B)=\dfrac{0.45}{0.75}=0.6$

(b) $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B) \quad\Rightarrow\quad 0.9=\mathrm{P}(A)+0.6-0.45 \quad\Rightarrow\quad \mathrm{P}(A)=0.75$

(c) A and B are independent events because $\mathrm{P}(A)=\mathrm{P}(A|B)$

3. [Maximum mark: 5]

Consider the function $y = p + \frac{p^2}{x} + x^2,\ x \neq 0$,where p is a constant.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [1]

(b) The graph of the function has a local minimum point at $(2,8)$. Find the value of p. [4]

Solution:

(a) $y = p + p^2x^{-1} + x^2 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -p^2x^{-2} + 2x = -\frac{p^2}{x^2} + 2x$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{p^2}{x^2} + 2x = 0$ when $x = 2$: $-\frac{p^2}{4} + 4 = 0 \quad \Rightarrow \quad p^2 = 16 \quad \Rightarrow \quad p = \pm 4$

$p = 4:\ y = 4 + \frac{16}{x} + x^2 \quad \Rightarrow \quad y(2) = 4 + \frac{16}{2} + 2^2 = 16$

$p = -4:\ y = -4 + \frac{16}{x} + x^2 \quad \Rightarrow \quad y(2) = -4 + \frac{16}{2} + 2^2 = 8$

Therefore, $p = -4$

4. [Maximum mark: 6]

A multiple choice test consists of twelve questions. Each question has four answers. Only one of the answers is correct. For each question, Emma randomly chooses one of the fours answers.

(a) Write down the expected number of questions Emma answers correctly. [1]

(b) Find the probability that Emma answers exactly four questions correctly. [2]

(c) Find the probability that Emma answers more than four questions correctly. [3]

Solution:

(a) $\mathrm{E}(X) = 12 \cdot \frac{1}{4} = 3$

(b) $X \sim \mathrm{B}\left(12, \frac{1}{4}\right);\quad \mathrm{P}(X = 4) = \binom{12}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^8 = 0.19358\ldots \approx 0.194$

or calculate with GDC: $\mathrm{binomPdf}\left(12, \frac{1}{4}, 4\right)$ 0.193577706814

(c) $\mathrm{P}(X > 4) = 1 - \left[\mathrm{P}(X=0) + \mathrm{P}(X=1) + \mathrm{P}(X=2) + \mathrm{P}(X=3) + \mathrm{P}(X=4)\right]$

$= 1 - [0.031676\ldots + 0.12671\ldots + 0.23229\ldots + 0.25810\ldots + 0.19358\ldots] = 1 - 0.84236\ldots \approx 0.158$

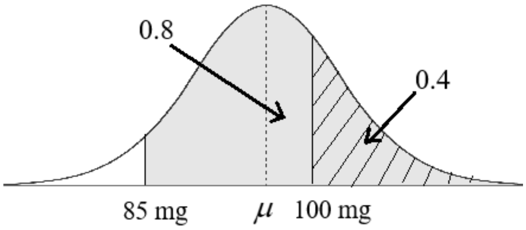
or calculate with GDC: $\mathrm{binomCdf}\left(12, \frac{1}{4}, 5, 12\right)$ 0.157643676073

5. [Maximum mark: 7]

It is known that two out of five cups of coffee served at Bella's Coffee Shop contain more than 100 mg of caffeine. It is also known that four out of five cups served at Bella's contain more 85 mg of caffeine.

Assuming the amount of caffeine in a cup of coffee at Bella's is modelled by a normal distribution, find the mean and standard deviation of the caffeine content in a cup of coffee served at Bella's.

Solution: $\mu = ?\quad \sigma = ?\qquad z = \dfrac{x-\mu}{\sigma}$



$$\mathrm{P}(X<85) = 1-\mathrm{P}(X>85) = 0.2 \quad\Rightarrow\quad z = -0.84162...$$

$$-0.84162... = \frac{85-\mu}{\sigma} \quad\Rightarrow\quad \mu + (-0.84162...)\sigma = 85$$

$$\mathrm{P}(X<100) = 1-\mathrm{P}(X>100) = 0.6 \quad\Rightarrow\quad z = 0.25335...$$

$$0.25335... = \frac{100-\mu}{\sigma} \quad\Rightarrow\quad \mu + (0.25335...)\sigma = 100$$

$$\begin{cases}\mu + (-0.84162...)\sigma = 85\\ \mu + (0.25335...)\sigma = 100\end{cases} \Rightarrow \text{ use linear equation solver on GDC } \Rightarrow \mu \approx 96.5 \text{ mg},\ \sigma = 13.7 \text{ mg}$$

invNorm(0.2,0,1)	-0.841621233465
-0.84162123346456→z1	-0.841621233465
invNorm(0.6,0,1)	0.253347101143
0.25334710114285→z2	0.253347101143

$\mu = m, \sigma = s$

$$\text{linSolve}\left(\begin{cases}m+z1\cdot s=85\\ m+z2\cdot s=100\end{cases},\{m,s\}\right)$$

$$\{96.5293914015,13.6990262877\}$$

6. [Maximum mark: 7]

Prove that the sum of the cubes of any two consecutive odd integers is divisible by four.

Solution:

Given $n \in \mathbb{Z}$ then $2n-1$ and $2n+1$ represent two consecutive odd integers.

$$(2n-1)^3 + (2n+1)^3 = 8n^3 - 12n^2 + 6n - 1 + 8n^3 + 12n^2 + 6n + 1$$

$$= 16n^3 + 12n = 4(4n^3 + 3n)$$

Thus, the sum of the cubes of any two consecutive odd integers is divisible by four.

7. [Maximum mark: 5]

In the quadratic equation $7x^2 - 8x + c = 0,\ c \in \mathbb{Q}$, one root is three times the other root. Find the exact value of c.

Solution:

Let α and 3α be the two roots $\Rightarrow\ \alpha + 3\alpha = -\dfrac{-8}{7}$ and $\alpha \cdot 3\alpha = \dfrac{c}{7}$

$$4\alpha = \frac{8}{7} \Rightarrow \alpha = \frac{2}{7};\ 3\alpha^2 = \frac{c}{7} \Rightarrow 3\left(\frac{2}{7}\right)^2 = \frac{c}{7} \Rightarrow \frac{12}{7^2} = \frac{c}{7} \Rightarrow c = \frac{12}{7}$$

8. [Maximum mark: 7]

Find the value of the constant term in the expansion $(x-1)^3\left(\frac{1}{2x^2}+4x\right)^6$.

Solution:

$$(x-1)^3\left(\left(2x^2\right)^{-1}+4x\right)^6=\left(x^3-3x^2+3x-1\right)\left[\sum_{r=0}^{6}\binom{6}{r}\left(\frac{1}{2}x^{-2}\right)^{6-r}(4x)^r\right]$$

$$=\left(x^3-3x^2+3x-1\right)\left(\_x^6+\_x^3+\_x^0+\_x^{-3}+\_x^{-6}+\_x^{-9}+\_x^{-12}\right)$$

hence, constant term $=x^3\left(\_x^{-3}\right)+(-1)\left(\_x^0\right)$

find x^{-3} term of $\sum_{r=0}^{6}\binom{6}{r}\left(\frac{1}{2}x^{-2}\right)^{6-r}(4x)^r$: exponent $=-2(6-r)+r=-3 \Rightarrow r=3$

$$\binom{6}{3}\left(\frac{1}{2}x^{-2}\right)^{6-3}(4x)^3=20\cdot\frac{1}{8}\cdot x^{-6}\cdot 64\cdot x^3=160x^{-3}$$

find x^0 term of $\sum_{r=0}^{6}\binom{6}{r}\left(\frac{1}{2}x^{-2}\right)^{6-r}(4x)^r$: exponent $=-2(6-r)+r=0 \Rightarrow r=4$

$$\binom{6}{4}\left(\frac{1}{2}x^{-2}\right)^{6-4}(4x)^4=15\cdot\frac{1}{4}\cdot x^{-4}\cdot 256\cdot x^4=960$$

thus, the constant term $=x^3\cdot 160x^{-3}+(-1)960=-800$

9. [Maximum mark: 6]

Show that $\arctan(x+2)-\arctan(x+1)=\arctan\left(\frac{1}{x^2+3x+3}\right)$.

Solution:

Apply the compound angle identity $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$ to the left side

$$\tan\left[\arctan(x+2)-\arctan(x+1)\right]=\frac{\tan\left[\arctan(x+2)\right]-\tan\left[\arctan(x+1)\right]}{1+\tan\left[\arctan(x+2)\right]\tan\left[\arctan(x+1)\right]}$$

$$\tan\left[\arctan(x+2)-\arctan(x+1)\right]=\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}$$

$$\tan\left[\arctan(x+2)-\arctan(x+1)\right]=\frac{1}{x^2+3x+3}$$

Now, take arctan of both sides

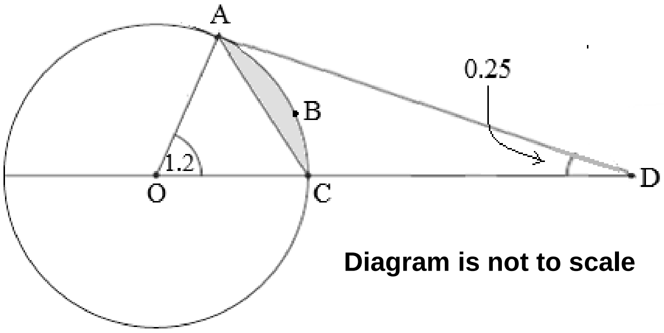
$$\arctan\left\{\tan\left[\arctan(x+2)-\arctan(x+1)\right]\right\}=\arctan\left(\frac{1}{x^2+3x+3}\right)$$

$$\arctan(x+2)-\arctan(x+1)=\arctan\left(\frac{1}{x^2+3x+3}\right)$$ ***Q.E.D.***

Section B (55 marks)

10. [Maximum mark: 18]

The diagram below shows a circle with centre O and radius 6 cm.



The points A, B and C lie on the circle. The point D is outside the circle and lies on (OC). Angle $\text{AOC} = 1.2$ radians and angle $\text{ADO} = 0.25$ radians.

(a) Find the area of the sector OABC. [3]

(b) Find the area of the shaded region bounded by the chord AC and the arc ABC. [4]

(c) Find AD. [3]

(d) Find OD [4]

(e) Find the area of the region ABCD. [4]

Solution:

(a) $A = \frac{1}{2}\theta r^2 \quad \Rightarrow \quad \frac{1}{2}(1.2)6^2 = 21.6 \text{ cm}^2$

(b) area of shaded region = area of sector OABC – area of triangle OAC

$$= 21.6 - \frac{1}{2}\cdot 6\cdot 6\cdot \sin(1.2) = 21.6 - 16.7767... \approx 4.82 \text{ cm}^2$$

(c) sine rule: $\frac{\text{OA}}{\sin(\text{ODA})} = \frac{\text{AD}}{\sin(\text{AOC})} \quad \Rightarrow \quad \frac{6}{\sin(0.25)} = \frac{\text{AD}}{\sin(1.2)} \quad \Rightarrow \quad \text{AD} \approx 22.6 \text{ cm}$

(d) $\text{OAD} = \pi - (1.2 + 0.25) \approx 1.69159...$

sine rule: $\frac{\text{OA}}{\sin(\text{ODA})} = \frac{\text{OD}}{\sin(\text{OAD})} \quad \Rightarrow \quad \frac{6}{\sin(0.25)} = \frac{\text{OD}}{\sin(1.69159...)} \quad \Rightarrow \quad \text{OD} \approx 24.1 \text{ cm}$

[**note**: use ‘full calculator accuracy’ values for AD and OD in calculations for part (e) below]

(e) area region ABCD = area triangle OAD – area sector OABC

area of triangle OAD $= \frac{1}{2}\cdot 6(24.0751...)\sin(1.2) \approx 67.3168... \text{cm}^2$

OR area triangle OAD $= \frac{1}{2}(22.6037...)(24.0751...)\sin(0.25) \approx 67.3168... \text{cm}^2$

thus, area of region ABCD $= 67.3168... - 21.6 \approx 45.7 \text{ cm}^2$

11. [Maximum mark: 21]

Consider the points $\mathrm{P}(2,-1,0)$, $\mathrm{Q}(3,0,1)$ and $\mathrm{R}(1,m,2)$, such that $m \in \mathbb{Z}, m<0$.

(a) (i) Find the scalar product $\vec{\mathrm{QP}} \cdot \vec{\mathrm{QR}}$.

(ii) Hence, given that $\mathrm{P\hat{Q}R} = \arccos \frac{\sqrt{2}}{3}$, show that $m=-1$. [6]

(b) Determine the Cartesian equation of the plane Π containing points P, Q and R. [4]

(c) Find the **exact** area of triangle PQR. [4]

(d) (i) The line L is perpendicular to plane Π and passes through P. Find a vector equation of L.

(ii) The point $S(6,-7,2)$ lies on L. Find the volume of the pyramid PQRS. [7]

Solution:

(a) (i) $\vec{\mathrm{QP}} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ $\qquad \vec{\mathrm{QR}} = \begin{pmatrix} 1 \\ m \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ m \\ 1 \end{pmatrix}$

$$\vec{\mathrm{QP}} \cdot \vec{\mathrm{QR}} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ m \\ 1 \end{pmatrix} = (-1)(-2) + (-1)m + (-1)(1) = 1-m$$

(ii) $\vec{\mathrm{QP}} \cdot \vec{\mathrm{QR}} = \left|\vec{\mathrm{QP}}\right| \left|\vec{\mathrm{QR}}\right| \cos \mathrm{P\hat{Q}R}$

$$1-m = \sqrt{3}\sqrt{m^2+5}\left(\frac{\sqrt{2}}{3}\right) \Rightarrow (1-m)^2 = \left[\sqrt{3}\sqrt{m^2+5}\left(\frac{\sqrt{2}}{3}\right)\right]^2$$

$$m^2-2m+1 = 3\left(m^2+5\right)\frac{2}{9} \Rightarrow \frac{m^2}{3} - 2m - \frac{7}{3} = 0 \Rightarrow m^2-6m-7=0 \Rightarrow (m-7)(m+1)=0$$

since given that $m<0$, then $m=-1$ ***Q.E.D.***

(b) normal vector for Π: $\boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -1 & -1 \\ -2 & -1 & 1 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} -1 & -1 \\ -2 & -1 \end{vmatrix} = -2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$

Cartesian equation of plane Π: $-2x+3y-z=d$

substitute in $\mathrm{P}(2,-1,0)$ to find d: $d=-2(2)+3(-1)-0=-7$

thus, Cartesian equation of plane Π is $-2x+3y-z=-7$ [or $2x-3y+z=7$]

[***continued on next page***]

(c) $\cos P\hat{Q}R = \frac{\sqrt{2}}{3} \Rightarrow \sin P\hat{Q}R = \sqrt{1-\left(\frac{\sqrt{2}}{3}\right)^2} = \sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3}$; $\left|\overrightarrow{QP}\right| = \sqrt{3}$, $\left|\overrightarrow{QR}\right| = \sqrt{6}$

area of triangle PQR $= \frac{1}{2}\left|\overrightarrow{QP}\right|\left|\overrightarrow{QR}\right|\sin P\hat{Q}R = \frac{1}{2}\sqrt{3}\sqrt{6}\frac{\sqrt{7}}{3} = \frac{\sqrt{126}}{6}$ units² $\left[\text{OR } \frac{\sqrt{9}\sqrt{14}}{6} = \frac{\sqrt{14}}{2}\right]$

(d) (i) if line L is perpendicular to plane Π, then L is parallel to normal vector of Π

hence, a vector equation for line L is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$

(ii) volume of a right pyramid $= \frac{1}{3}Ah$ where A is area of base and h is height; $A = \frac{\sqrt{14}}{2}$, $h = \left|\overrightarrow{PS}\right|$

$\overrightarrow{PS} = \begin{pmatrix} 6 \\ -7 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}$ $h = \left|\overrightarrow{PS}\right| = \sqrt{4^2 + (-6)^2 + 2^2} = \sqrt{56} = 2\sqrt{14}$

volume of pyramid PQRS $= \frac{1}{3}\cdot\frac{\sqrt{14}}{2}\cdot 2\sqrt{14} = \frac{14}{3} \approx 4.67$ units³

12. [Maximum mark: 16]

Consider the function $f(x) = \ln(1+\sin x)$. The Maclaurin series for $f(x)$ up to and including the x^4 term is $f(x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$.

(a) Show that the Maclaurin series for $g(x) = \ln(1-\sin x)$ up to and including the x^4 term is $g(x) = -x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \cdots$. [3]

(b) Use the Maclaurin series for $f(x)$ and $g(x)$ to show that the Maclaurin series for $h(x) = \ln(\cos x)$ up to and including the x^4 term is $h(x) = -\frac{x^2}{2} - \frac{x^4}{12} + \cdots$. [5]

(c) Hence, or otherwise, find the first two terms of the Maclaurin series for $q(x) = \tan x$. [4]

(d) Hence, calculate analytically (no GDC) the **exact** value of $\lim_{x\to 0}\left(\frac{\tan(x^2)}{\ln(\cos x)}\right)$. [4]

[***worked solution on next page***]

Solution:

(a) $f(x) = \ln(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$

$g(x) = \ln(1-\sin x) = \ln(1+\sin(-x)) = f(-x)$

Hence, $g(x) = \ln(1-\sin x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{6} - \frac{(-x)^4}{12} + \cdots = -x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \cdots$ ***Q.E.D.***

(b) $2\ln(\cos x) = \ln(\cos^2 x)$

$= \ln(1-\sin^2 x)$

$= \ln[(1+\sin x)(1-\sin x)]$

$= \ln(1+\sin x) + \ln(1-\sin x)$

$= f(x) + g(x)$

$= \left(x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots\right) + \left(-x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \cdots\right)$

$2\ln(\cos x) = -x^2 - \frac{x^4}{6} + \cdots$ thus, $h(x) = \ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} + \cdots$ ***Q.E.D.***

(c) $\frac{\mathrm{d}}{\mathrm{d}x}[\ln(\cos x)] = \frac{1}{\cos x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = \frac{-\sin x}{\cos x} = -\tan x$

hence, $\tan x = -\frac{\mathrm{d}}{\mathrm{d}x}[\ln(\cos x)]$

$= -\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{x^2}{2} - \frac{x^4}{12} + \cdots\right]$

$= x + \frac{x^3}{3} + \cdots$

(d) $\frac{\tan(x^2)}{\ln(\cos x)} = \frac{x^2 + \frac{(x^2)^3}{3} + \cdots}{-\frac{x^2}{2} - \frac{x^4}{12} + \cdots} = \frac{x^2 + \frac{x^6}{3} + \cdots}{-\frac{x^2}{2} - \frac{x^4}{12} + \cdots} = \frac{1 + \frac{x^4}{3} + \cdots}{-1 - \frac{x^2}{12} + \cdots}$

$\lim_{x\to 0}\left(\frac{\tan(x^2)}{\ln(\cos x)}\right) = \lim_{x\to 0}\left(\frac{1 + \frac{x^4}{3} + \cdots}{-\frac{1}{2} - \frac{x^2}{12} + \cdots}\right) = \frac{1}{-\frac{1}{2}} = -2$